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EIGENVALUES AND EXPANDERS

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Linear expanders have numerous applications to theoretical computer science. Here we show that a regular bipartite graph is an expander *if and only if* the second largest eigenvalue of its adjacency matrix is well separated from the first. This result, which has an analytic analogue for Riemannian manifolds enables one to generate expanders randomly and check efficiently their expanding properties. It also supplies an efficient algorithm for approximating the expanding properties of a graph. The exact determination of these properties is known to be coNP-complete.

1. Introduction

Let $G=(V, E)$ be a graph. For a subset X of V put

$$N(X) = \{v\in V\colon vx\in E \quad \text{for some} \quad x\in X\}.$$

An (n, d, c)-*expander* is a bipartite graph on the sets of vertices I (inputs) and O (outputs), where $|I|=|O|=n$, the maximal degree of a vertex is d, and for every set $X\subseteq I$ of cardinality $|X|=\alpha\leqq n/2$,

$$|N(X)| \geqq \big(1+c(1-\alpha/n)\big)\cdot \alpha. \tag{1.1}$$

It is a *strong* (n, d, c)-*expander* if (1.1) holds for all $X\subseteq I$. A family of linear expanders of *density* d and *expansion* c is a set $\{G_i\}_{i=1}^{\infty}$, where G_i is an (n_i, d, c)-expander, $n_i\to\infty$ and $n_{i+1}/n_i\to 1$ as $i\to\infty$.

Such families are the subject of an extensive literature. They form the main component in the recent parallel sorting network of Ajtai, Komlós and Szemerédi [2]. They also form the basic building block used in the construction of graphs with special connectivity properties and small number of edges (see, e.g. [13]). An example of a graph of this type is an n-*superconcentrator* (s. c.) which is a directed acyclic graph with n inputs and n outputs such that for every $1\leqq r\leqq n$ and every two sets A of r inputs and B of r outputs there are r vertex disjoint paths from the vertices of A to the vertices of B. A family of linear s.c.-s of density k is a set $\{G_n\}_{n=1}^{\infty}$ where G_n is an n-s.c. with $\leqq(k+o(1))n$ edges. Superconcentrators are relevant to computer science in

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AMS subject classification (1980): 05 C 99, 05 C 50, 68 E 10 several ways. They have been used in the construction of graphs that are hard to pebble (see [21], [26], [27]), in the study of lower bounds ([32]) and in the establishment of time space tradeoffs for computing various functions ([1], [19], [31]).

It is not too difficult to prove the existence of a family of strong linear expanders (and hence of a family of linear s.c.-s) using the so-called "probabilistic construction". In fact, one can show that almost every graph in a properly chosen class of graphs (e.g., the class of all d-regular bipartite graphs on n inputs and n outputs, for $d \geqq 3$) is a strong (n, d, c)-expander for some $c=c(d)$. (See, e.g. [7], [13], [24], [25]). However, an explicit construction is much more difficult. The first such construction was given by Margulis [22] and improved in [17], where a family of strong linear expanders of density 7 and expansion $(2-\sqrt{3})/2$ is explicitly described and used to construct a family of linear s.c.-s of density $\simeq 271.8$. See also [3, 4] for a more general construction and for better s.c.-s.

The explicit construction is, however, a rather poor substitute for the probabilistic one. Thus, for example, almost every 7-regular bipartite graph has better expansion properties than the one constructed in [17]. Hence, it has been suggested by several authors (cf. [9]) to generate expanders randomly, and then check if the generated graphs have the desired expansion properties. However, as proved in [9], even the problem of checking if a given graph is an $(n, d, 0)$-expander is coNP-complete. Thus, the random generation method seems to be impractical.

Our main result here implies that this method is, in fact, practical. We prove a very close relationship between $\lambda(G)$ — the second smallest eigenvalue of a certain matrix associated with a graph G — and its expansion properties. For example we show that a regular graph G is a strong expander if and only if $\lambda(G)$ is well separated from 0. The "if" part is not too difficult and is somewhat similar to a result of Tanner [30]. The "only if" part is much trickier and is the discrete analogue of a theorem of Cheeger [12] on Riemannian manifolds. In fact, we can prove this part (at least for 3-regular graphs) by associating, as in [11], Riemannian manifolds with graphs and by using Cheeger's theorem. However, we prefer to give here a direct elementary proof that requires only combinatorial reasoning and linear algebra. Since there are several efficient algorithms to compute eigenvalues of matrices (see, e.g. [28]), one can really apply our results to generate expanders randomly and then check their expansion properties.

Our methods are fruitful not only for generating expanders randomly. We also show here how they supply an efficient way of generating graphs with higher amount of expansion from expanders with weaker expansion properties. In [3] we combine similar methods with results of Kazhdan on group representations to obtain many new examples of strong linear expanders. In [5] we apply these methods together with the Fourier analysis method of [17] and the methods of [20] to obtain better explicit expanders than those previously known which enable us to construct an explicit family of linear s.c.-s of density asymptotic to 122.74, better than the previous known constructions. Finally, we have recently found a way of applying our methods to the problem of designing fault tolerant processor arrays, discussed in [29] . This will appear somewhere else.

For our purposes, it seems convenient to deal with *magnifiers*, the nonbipartite analogues of expanders. These are defined in Section 2, where we also establish the close relationship between eigenvalues and magnifiers. In Section 3 we observe that this relationship implies a similar one between eigenvalues and expanders. In Section 4 we show how to generate expanders (and magnifiers) randomly. As a byproduct we obtain some new results on eigenvalues of random regular graphs. Section 5 contains some concluding remarks and open problems.

2. Magnifiers, eigenvalues and enlargers

An *(n, d, c)-magnifier* is a graph $G=(V, E)$ on n vertices, with maximal degree d such that for every set $X\subseteq V$ that satisfies $|X|\leqq n/2$, $|N(X)-X|\geqq c\cdot|X|$ holds. The *(extended) double cover* of a graph $G=(V, E)$, where $V=\{v_1, v_2, \dots, v_n\}$ is the bipartite graph H on the sets of inputs $X=\{x_1, \dots, x_n\}$ and outputs $Y=\{y_1, \dots, y_n\}$ in which $x_i\in X$ and $y_j\in Y$ are adjacent iff $i=j$ or $v_iv_j\in E$. (Notice that this is the usual double cover of G plus the perfect matching $(x_i\, y_i)_{i=1}^n$.)

The following obvious lemma reveals the tight connection between magnifiers and expanders.

Lemma 2.1. *The double cover of an (n, d, c)-magnifier is an $(n, d+1, c)$-expander.* ∎

In this section we show the close relation between eigenvalues and magnifiers. Let $G=(V, E)$ be a graph, $|V|=n$. The *adjacency matrix* $A_G=(a_{uv})_{u\in V, v\in V}$ of G is a $0-1$ matrix where $a_{uv}=1$ iff $uv\in E$. Put $Q_G=\operatorname{diag}(d(v))_{v\in V}-A_G$, where $d(v)$ is the degree of the vertex $v\in V$. Let $\lambda_0\leqq\lambda_1=\lambda(G)\leqq\lambda_2\leqq\dots\leqq\lambda_{n-1}$ be the eigenvalues of Q_G each appearing according to its multiplicity. One can easily check that $\lambda_0=0$ and the constant vector is its corresponding eigenvector. Moreover, $\lambda_1=\lambda(G)\geqq0$ with equality iff G is not connected. The matrix Q_G is commonly used in graph theory in finding the number of spanning trees of G (see, e.g., [8, Ch. 6]) and its spectrum was investigated by various authors ([3], [6], [14], [15]). It is, in some sense, the discrete analogue of the Laplace operator and the main results we obtain here have analytic analogues to Riemannian manifolds. Here we restrict our attention to $\lambda=\lambda(G)$, which turns out to describe the expansion properties of G. An *(n, d, ε)-enlarger* is a graph on n vertices with maximal degree d and $\lambda\geqq\varepsilon$.

The following lemma is proved by Milman and the present author in [3] using elementary linear algebra.

Lemma 2.2. ([3, *Theorem* 2.5]) *Let $G=(V, E)$ be a graph on n vertices and put $\lambda=\lambda(G)$. Suppose $A, B\subseteq V$ are disjoint sets of vertices and let $\varrho=\varrho(A, B)>1$ be the distance between them. If d is the maximal degree of a vertex of G, $a=|A|/n$ and $b=|B|/n$ then*

$$b\leqq(1-a)\Big/\left(1+\frac{\lambda}{d}a\varrho^2\right). \quad ∎$$

(Actually we can prove a slightly stronger result, but we omit it to avoid too complicated statements.)

Corollary 2.3. *Every (n, d, ε)-enlarger is an (n, d, c)-magnifier, where $c=2\varepsilon/(d+2\varepsilon)$.*

Proof. Let $G=(V, E)$ be an (n, d, ε)-enlarger. Suppose $X\subseteq V$, $|X|\leqq n/2$. We have to show that $|N(X)-X|\geqq c|X|$. By Lemma 2.2, with $A=X$, $B=V-(X\cup N(X))$,

$\varrho=\varrho(A,B)\geqq 2$ and $\lambda=\lambda(G)\geqq\varepsilon$ we obtain

$$1-\frac{|X|+|N(X)-X|}{n}\leqq\left(1-\frac{|X|}{n}\right)\Big/\left(1+\frac{\varepsilon}{d}\frac{|X|}{n}\cdot 2^2\right).$$

Thus

$$|N(X)-X|\geqq\frac{4\varepsilon}{d+4\varepsilon\frac{|X|}{n}}\left(1-\frac{|X|}{n}\right)\cdot|X|$$

and for $|X|\leqq n/2$ this implies $|N(X)-X|\geqq\frac{2\varepsilon}{d+2\varepsilon}|X|=c|X|$, as needed. ∎

The last corollary states that every enlarger is a magnifier. Our next lemma, which is the main result of this section, shows that the converse is also true, namely, that a magnifying graph G has a relatively large $\lambda=\lambda(G)$.

Lemma 2.4. *Let $G=(V,E)$ be an (n,d,c) magnifier. Then $\lambda(G)\geqq c^2/(4+2c^2)$, i.e., G is an (n,d,ε)-enlarger, where $\varepsilon=c^2/(4+2c^2)$.*

Proof. Recall that $\lambda=\lambda(G)$ is the second smallest eigenvalue of the symmetric matrix $Q=Q_G=\operatorname{diag}(d(v))-A_G$. Let $f\colon V\to\mathbf{R}$ be the corresponding eigenvector. Since the eigenvectors of Q are orthogonal and the eigenvector of the 0-eigenvalue is $(1,1,\dots,1)$;

$$\sum_{v\in V}f(v)=0,\quad(\text{and } f\not\equiv 0).$$

Put $V^+=\{v\in V\colon f(v)>0\}$, $V^-=V-V^+$. Without loss of generality we can assume that $0<|V^+|\leqq n/2$ (otherwise replace f by $-f$). Let $E(V^+,V^+)$ denote the set of all edges $uv\in E$ where $u,v\in V^+$. Similarly put $E(V^+,V^-)=\{uv\in E\colon u\in V^+,\ v\in V^-\}$. We also define $g\colon V\to\mathbf{R}$ by

$$g(v)=\begin{cases}f(v) & \text{if } v\in V^+\\ 0 & \text{otherwise.}\end{cases}$$

By the definitions of λ, f: $(Qf)(v)=\lambda f(v)$ for all $v\in V$. Hence

$$\lambda=\frac{\sum_{v\in V^+}(Qf)(v)\cdot f(v)}{\sum_{v\in V^+}f^2(v)}.$$

However,

$$\sum_{v\in V^+}(Qf)(v)\cdot f(v)=\sum_{v\in V^+}\Big(d(v)f^2(v)-\sum_{u\in N(v)}f(v)f(u)\Big)=$$

$$=\sum_{uv\in E(V^+,V^+)}(f(u)-f(v))^2+\sum_{uv\in E(V^+,V^-)}f(u)\cdot(f(u)-f(v))\geqq$$

$$\geqq\sum_{uv\in E}(g(u)-g(v))^2.$$

Clearly

$$\sum_{v\in V^+}f^2(v)=\sum_{v\in V^+}g^2(v)=\sum_{v\in V}g^2(v).$$

Thus

$$\lambda \geqq \frac{\sum_{uv\in E}(g(u)-g(v))^2}{\sum_{v\in V} g^2(v).}. \tag{2.1}$$

We now show that the magnifying properties of G supply a lower bound to the right-hand side of inequality (2.1). This is done using the max-flow min-cut theorem. Consider the network N with the vertex set $\{s, t\}\cup X\cup Y$ where s is the source, t is the sink and $X=V^+$, $Y=V$ are disjoint sets of vertices. The arcs of our network and their capacities are given by:

(a) For every $u\in X$ the arc (s, u) has capacity $1+c$.
(b) For every $u\in X$, $v\in Y$ the arc (u, v) has capacity 1 if $uv\in E$ or $u=v$ and 0 otherwise.
(c) For every $v\in Y$ the arc (v, t) has capacity 1.

We claim that the value of the min-cut of this network is $(1+c)\cdot|V^+|$. Indeed, the cut consisting of all the (s, u) arcs $(u\in V^+)$ has this capacity. For any other cut C, put $U=\{u\in V^+;\ C$ does not contain the arc $(s, u)\}$. The set of neighbours of U, $N(U)$ in Y, satisfies by the magnifying properties of our graph, $|N(U)|\geqq(1+c)|U|$. For every $v\in N(U)$ our cut must contain some arc incident with v, and all these arcs are distinct and have capacity 1. Hence the total capacity of C is $\geqq(1+c)|V^+-U|+ +|N(U)|\geqq(1+c)|V^+|$ proving our claim. By the max-flow min-cut theorem there exists an orientation $\bar{E}$ of the set of edges E of G and a function $h\colon \bar{E}\to\mathbf{R}$ such that:

$$0\leqq h(u, v)\leqq 1 \quad \text{for all} \quad (u, v)\in\bar{E}. \tag{i}$$

$$\sum_{v;\,(u,v)\in\bar{E}} h(u, v) = \begin{cases}1+c & \text{if} \quad u\in V^+\\ 0 & \text{otherwise}\end{cases} \tag{ii}$$

$$\sum_{u;\,(u,v)\in\bar{E}} h(u, v)\leqq 1 \quad \text{for all} \quad v\in V. \tag{iii}$$

It is easy to check that if $g\colon V\to\mathbf{R}$ is the function defined above then

$$\begin{aligned}\sum_{(u,v)\in\bar{E}} h^2(u, v)(g(u)+g(v))^2 &\leqq 2\sum_{(u,v)\in\bar{E}} h^2(u, v)\cdot(g^2(u)+g^2(v))\\ &= 2\sum_{u\in V} g^2(u)\cdot\Big(\sum_{v;\,(u,v)\in\bar{E}} h^2(u, v)+\sum_{v;\,(v,u)\in\bar{E}} h^2(v, u)\Big)\leqq\\ &\leqq 2(2+c^2)\sum_{u\in V} g^2(u),\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\sum_{(u,v)\in\bar{E}} h(u, v)\cdot(g^2(u)-g^2(v)) &= \sum_{u\in V} g^2(u)\cdot\Big(\sum_{v;\,(u,v)\in\bar{E}} h(u, v)-\sum_{v;\,(v,u)\in\bar{E}} h(v, u)\Big)\geqq\\ &\geqq c\sum_{u\in V} g^2(u).\end{aligned} \tag{2.3}$$

Combining inequalities (2.1), (2.2) and (2.3) with the Cauchy—Schwarz inequality we conclude that

$$\lambda \geqq \frac{\sum_{uv\in E} (g(u)-g(v))^2}{\sum_{v\in V} g^2(v)}$$

$$= \frac{\sum_{(u,v)\in E} (g(u)-g(v))^2 \cdot \sum_{(u,v)\in E} h^2(u,v)\big(g(u)+g(v)\big)^2}{\sum_{v\in V} g^2(v) \cdot \sum_{(u,v)\in E} h^2(u,v)\big(g(u)+g(v)\big)^2}$$

$$\geqq \frac{\Big(\sum_{(u,v)\in E} h(u,v)|g^2(u)-g^2(v)|\Big)^2}{2\cdot(2+c^2)\cdot\Big(\sum_{v\in V} g^2(v)\Big)^2}$$

$$\geqq \frac{1}{4+2c^2}\cdot\left(\frac{\sum_{(u,v)\in E} h(u,v)\big(g^2(u)-g^2(v)\big)}{\sum_{v\in V} g^2(v)}\right)^2 \geqq c^2/(4+2c^2).$$

This completes the proof. ▮

As mentioned above, the matrix Q_G is the discrete analogue of the Laplace operator. There is a close relationship between Lemma 2.4 and its analytic analogue proved by Cheeger in [12]. However, the discrete version appears to be somewhat more complicated, since we do not deal here with continuous functions. This also seems to be the reason for the difference between the discrete estimate $\big(c^2/(4+2c^2)\big)$ and the analytic one $(c^2/4)$. We can, in fact, prove a somewhat weaker version of Lemma 2.4 (at least for 3-regular graphs), by associating, as in [11], Riemannian manifolds with graphs and by using Cheeger's theorem.

It is also worth noting that the analytic analogue of Lemma 2.2 was proved in [18]. Here the main difference between the discrete version and its analytic analogue is the factor d corresponding to the maximum degree of a vertex of G. This difference seems to arise from the fact that there is no discrete analogue to the unique direction of the gradient.

The content of Corollary 2.3 and Lemma 2.4 is summarized in the following theorem.

Theorem 2.5. *Every (n, d, ε)-enlarger is an (n, d, c)-magnifier, where $c=2\varepsilon/(d+2\varepsilon)$. Every (n, d, c)-magnifier is an (n, d, ε)-enlarger, where $\varepsilon=c^2/(4+2c^2)$. Thus ,if a graph is an (n, d, c)-magnifier, one can prove efficiently (by computing eigenvalues) that it is an (n, d, c')-magnifier, where $c'=c^2/\big(c^2+d(2+c^2)\big)$.* ▮

We conclude this section by showing how to apply Lemma 2.2 in order to generate graphs with (provably) strong magnifying properties from a graph G with a relatively small eigenvalue $\lambda=\lambda(G)$. The idea is the well known "iterated" construction mentioned already in [22], but the estimate is better.

Let $G=(V, E)$ be a graph on a set V of n vertices with maximal degree d. Put $\lambda=\lambda(G)$ and suppose λ is relatively small compared to d (which is the case in the known constructions of magnifiers). Let $X \subseteq V$ be a set of vertices with $|X| \leqq n/2$. By Corollary 2.3

$$|X \cup N(X)| \geqq \left(1+\frac{2\lambda}{d+2\lambda}\right)|X|. \tag{2.4}$$

Suppose one wants stronger magnifying properties. For example suppose that a graph H is needed such that for every $X \subseteq V$, $|X| \leqq n/2e$ the inequality $|N_H(X) \cup X| \geqq e|X|$ will hold. Let H be the graph on the set of vertices V in which $u, v \in V$ are joined if and only if their distance in G is $\leqq k = 2+d/(2\lambda)$. One can easily check, by applying (2.4) repeatedly, that H has the desired magnifying properties. The maximal degree of H can be, however, $d(1+(d-1)+\ldots+(d-1)^{k-1})$. In order to keep this degree as small as possible it is desirable to choose smaller values of k. The next result shows that, in fact, for the above example $k = O(\sqrt{d/\lambda}) \ll 2+d/(2\lambda)$ suffices. This result can also be deduced from Theorem 2.5 of [3].

Proposition 2.6. *Let $G=(V,E)$ be a graph on n vertices with maximal degree d and $\lambda = \lambda(G)$. Let H be the graph on the set of vertices V in which $u, v \in V$ are adjacent iff the distance between them in G is $\leqq k$. Then H is an (n, d', c')-magnifier where $d' \leqq d(1+\ldots+(d-1)^{k-1})$ and*

$$c' = \frac{1}{2}\lambda(k+1)^2 \Big/ \left(d+\frac{1}{2}\lambda(k+1)^2\right).$$

Proof. The proof is analogous to that of Corollary 2.3. Suppose $X \subseteq V$, $|X| \leqq n/2$. We have to show that $|N_H(X)-X| \geqq c'|X|$. Put $B = V-(X \cup N_H(X))$. Note that the distance between X and B in G is $\geqq k+1$. By Lemma 2.2

$$1-\frac{|X|+|N(X)-X|}{n} \leqq \left(1-\frac{|X|}{n}\right)\Big/\left(1+\frac{\varepsilon}{d}\frac{|X|}{n}(k+1)^2\right)$$

and the desired result follows. ∎

By the above proposition if e.g., $k=\sqrt{2d/\lambda}$ then for all $|X| \leqq n/2$ $|N_H(X) \cup X| \geqq 3|X|/2$. If a higher amount of expansion is needed we may use the previous argument to estimate the magnifying properties of "iterates" of H. Thus, e.g., $k = 2\sqrt{2d/\lambda}$ is enough to guarantee expansion of the form $|N_H(X) \cup X| \geqq (9/4)|X|$ for all $|X| \leqq n/3$. This construction, together with Lemma 2.1, seems useful, e.g., in obtaining explicit expanders for the sorting network of [2], in which a relatively high amount of expansion is needed.

3. Eigenvalues, expanders and magnifiers

A connection between expanders and magnifiers has already been established in Lemma 2.1. As mentioned in the introduction, magnifiers are in a sense, the non-bipartite generalization of expanders. The following lemmas show that every strong expander is a magnifier and that every regular bipartite magnifier is a (strong) expander. These, together with Theorem 2.5 establish a close relationship between the expanding properties of a regular bipartite graph G and the value of $\lambda(G)$, and will enable us to generate random expanders, and check efficiently their expanding properties.

Lemma 3.1. *Let $G=(I, O; E)$ be an (n, d, c) strong expander. Then G is a $(2n, d, c/16)$-magnifier.*

Remark. It is worth noting that the constant 1/16 can easily be improved. We made no attempt to find the best possible constant. Also note that without the word "strong" the conclusion of Lemma 3.1 does not hold. In fact, an (n, d, c) expander $(c>0)$ need not even be connected and thus, certainly must not be a magnifier. An example of a disconnected $(n, n-1, 1)$-expander is the disjoint union of $K_{1,n/4}$ and $K_{n-1,3n/4}$, where $K_{a,b}$ denotes the complete bipartite graph with classes of vertices of sizes a and b. For every $c'>0$ this is *not* a $(2n, n-1, c')$ magnifier since if X is the set of vertices of the $K_{1,n/4}$, then $|N(X)-X|=0$. A similar example with bounded maximal degree d (independent of n) can also be given.

Proof of Lemma 3.1. By definition every $X_1 \subseteqq I$ satisfies

$$|N(X_1)| \geqq |X_1|+c\left(1-\frac{|X_1|}{n}\right)\cdot|X_1|. \tag{3.1}$$

Applying (3.1) to a set of size $n/2$ we conclude that $c \leqq 2$. For every $X_2 \subseteqq O$, applying (3.1) to the set $I-N(X_2)$ we conclude that $|N(X_2)| \geqq |X_2|$. Suppose $X \subset I \cup O$ satisfies $|X| \leqq n$. To prove our lemma we must show that

$$|N(X)-X| \geqq \frac{c}{16}|X|.$$

Put $X_1=X\cap I$, $X_2=X\cap O$ and consider the following three possible cases:

Case 1. $|X_1| \leqq |X_2|\cdot(1-c/8)$.

In this case $|N(X)-X| \geqq |N(X_2)|-|X_1| \geqq |X_2|-|X_1| \geqq \frac{c}{8}|X_2| \geqq \frac{c}{16}|X|$.

Case 2. $|X_2|\left(1-\frac{c}{8}\right)<|X_1| \leqq n/2$.

In this case

$$\begin{aligned}|N(X)-X| &\geqq |N(X_1)|-|X_2| \\ &\geqq \left(\left(1+\frac{c}{2}\right)-\frac{1}{1-c/8}\right)|X_1| = \frac{\frac{3}{8}c-\frac{c^2}{16}}{1-c/8}|X_1| \geqq \frac{\frac{3}{8}c-\frac{1}{8}c}{\left(2-\frac{c}{8}\right)}|X| \\ &\geqq \frac{c}{8}|X| \geqq \frac{c}{16}|X|.\end{aligned}$$

Case 3. $|X_1| \geqq n/2$.

In this case

$$|N(X)-X| \geqq |N(X_1)|-|X_2| \geqq \left(1+\frac{c}{2}\right)\frac{n}{2}-\frac{n}{2} = \frac{c}{4}\cdot n \geqq \frac{c}{4}|X| > \frac{c}{16}|X|.$$

This completes the proof. ∎

Next we show that a bipartite regular magnifier is a strong expander. Note that this is false without the regularity assumption. Indeed if $G=(I, O; E)$ where $I=\{i_1, \dots, i_n\}$, $O=\{o_1, o_2, \dots, o_n\}$ and

$$E = \{(i_j, o_j)\colon 1 \leqq j \leqq n/2\} \cup \{(i_m, o_l)\colon 1 \leqq m \leqq n, n/2 < l \leqq n\}$$

then G is a $(2n, n, 1/2)$ bipartite magnifier and for every $c>0$ G is not an (n, n, c) expander (since $|N(\{i_j : j>n/2\})|=|\{i_j : j>n/2\}|$.) A similar example with bounded maximal degrees (independent of n) can be given.

Lemma 3.2. *Let $G=(I, O; E)$ be a d-regular bipartite graph with $|I|=|O|=n$. If G is a $(2n, d, c)$-magnifier then G is an (n, d, b)-strong expander where $b=2c/((d+1)(c+1))$.*

Proof. Suppose $X \subseteq I$. We must show that

$$|N(X)| \geqq |X|+b\left(1-\frac{|X|}{n}\right)\cdot|X|. \tag{3.2}$$

Put $Y=N(X)$, $w=|Y|-|X|$ and consider the following two possible cases.

Case 1. $|X\cup Y|=|X|+|Y|\leqq n$.

By the magnifying properties of G, $|N(X\cup Y)-(X\cup Y)|=|N(Y)-X|\geqq c(|X|+|Y|)=c(2|X|+w)$. Note that there are precisely $d\cdot|X|$ edges joining vertices of X to vertices of Y, and hence precisely $d(|Y|-|X|)=dw$ edges joining vertices of Y to vertices in $N(Y)-X$. Thus $|N(Y)-X|\leqq dw$. Combining these inequalities we conclude that $dw\geqq c(2|X|+w)$, i.e., $w=|N(X)|-|X|\geqq\frac{2c}{d-c}|X|\geqq\frac{2c}{(d+1)(c+1)}|X|$ implying (3.2).

Case 2. $|X|+|Y|>n$.

Put $Z=N(Y)-X$, $z=|Z|$. As in Case 1,

$$z \leqq d(|Y|-|X|) = dw. \tag{3.3}$$

Define $T=(I\cup O)-(Z\cup X\cup Y)$. Clearly $|T|\leqq n$ and one can easily check that $N(T)-T\subseteq Z$. Thus, by the magnifying properties of G,

$$z = |Z| \geqq |N(T)-T| \geqq c|T| = c(2n-|X|-|Y|-|Z|) = c(2n-2|X|-w-z).$$

This implies

$$z \geqq \frac{c}{c+1}(2n-2|X|-w).$$

Combining the last inequality with (3.3) we obtain

$$dw \geqq \frac{c}{c+1}(2n-2|X|-w)$$

i.e.,

$$w = |N(X)|-|X| \geqq \frac{2c}{(d+1)(c+1)}(n-|X|) \geqq \frac{2c}{(d+1)(c+1)}\left(1-\frac{|X|}{n}\right)\cdot|X|,$$

implying (3.2). This completes the proof. ∎

Lemma 3.2 and Corollary 2.3 imply that if for a regular bipartite graph G, $\lambda(G)$ is well separated from 0, then G is a strong expander. One can derive a similar result with a somewhat better estimate from the main result of Tanner in [30]. We proceed to do this in the next lemma, which will be used in Section 4.

Lemma 3.3. *Let $G=(I, O; E)$ be a d-regular bipartite graph, where $|I|=|O|=n$. Put $\lambda=\lambda(G)$. Then G is an (n, d, c) strong expander, where*

$$c = (2d\lambda-\lambda^2)/d^2.$$

(One can easily show that always $\lambda \leqq d$ and hence $c \geqq \lambda/d$).

Proof. Let $c=(c_{io})_{i\in I, o\in O}$ be the $n\times n$ binary matrix whose rows and columns are indexed by the vertices of I and O, respectively, in which

$$c_{io} = \begin{cases} 1 & \text{if } io\in E \\ 0 & \text{otherwise.} \end{cases}$$

One can easily check that if $Q=Q_G$ is the Q-matrix of G defined in Section 2, then

$$(dI-Q)^2 = \begin{bmatrix} CC^T & 0 \\ 0 & C^TC \end{bmatrix}$$

and that if λ' is an eigenvalue of $dI-Q$ so is $-\lambda'$. Hence the two largest and two smallest eigenvalues of $dI-Q$ are $\pm(d, d-\lambda)$ and thus the two largest eigenvalues of C^TC are d^2 and $(d-\lambda)^2$. Therefore, by [30, Theorem 2.1], if $X\subseteqq I$ and $\alpha= =|X|/n$ then

$$|N(X)| \geqq \frac{d^2}{\alpha\big(d^2-(d-\lambda)^2\big)+(d-\lambda)^2}|X| = \left(1+\frac{(2d\lambda-\lambda^2)(1-\alpha)}{d^2-(2d\lambda-\lambda^2)(1-\alpha)}\right)|X|$$

$$\geqq \left(1+\frac{(2d\lambda-\lambda^2)}{d^2}\right)\left(1-\frac{|X|}{n}\right)\cdot|X|.$$

This completes the proof. Note that if $X\subseteqq I$, $|X|\leqq n/2$, we actually get

$$|N(X)| \geqq \left(1+\frac{2d\lambda-\lambda^2}{d^2-d\lambda+\lambda^2/2}\right)\left(1-\frac{|X|}{n}\right)\cdot|X|. \quad \blacksquare$$

Combining Lemmas 2.4, 3.1 and 3.3 we obtain the following result, analogous to Theorem 2.5.

Theorem 3.4. *Let $G=(I, O; E)$ be a d-regular bipartite graph, where $|I|=|O|=n$ and $\lambda=\lambda(G)$.*

(1) If G is an (n, d, c) strong expander then $\lambda\geqq c^2/(1024+2c^2)$ (i.e., G is an $(2n, d, c^2/(1024+2c^2))$-enlarger).

(2) If $\lambda\geqq\varepsilon$ (i.e., if G is a $(2n, d, \varepsilon)$-enlarger), then G is an $(n, d, (2d\varepsilon-\varepsilon^2)/d^2)$ strong expander.

Thus if G is an (n, d, c) strong expander, one can prove efficiently (by computing eigenvalues) that it is an (n, d, c') strong expander, where

$$c' = \frac{1}{d^2}\left(2d\frac{c^2}{1024+2c^2} - \frac{c^4}{(1024+2c^2)^2}\right) \geqq c^2/(1032d). \quad \blacksquare$$

The last inequality, in which we use the fact that $c \leqq 2$, can easily be improved. We will be more careful with the constant in the next section, where we generate random expanders.

4. Generating random expanders and magnifiers

In view of Theorems 2.5 and 3.4 and the well known fact that almost every graph in a properly chosen class of graphs has strong expanding properties (see, e.g., [7], [13], [25]), it is clear that one can generate graphs randomly and check efficiently that they have the desired properties. We proceed to describe the expected expanding properties of such graphs.

We will need the following result of Bassalygo [7]. (See also [13] for a similar result.)

Lemma 4.1. *Suppose* $0<\alpha<1/\beta<1$. *Let d be an integer satisfying*

$$d > \frac{H(\alpha)+H(\alpha\beta)}{H(\alpha)-\alpha\beta H(1/\beta)}, \tag{4.1}$$

where $H(X) = -X\log_2 X-(1-X)\log_2(1-X)$ *is the binary entropy function. Let I and O be two sets of vertices,* $|I|=|O|=n$, *and let G be a random d-regular bipartite graph on the classes of vertices I and O, obtained by choosing randomly d permutations from I to O. Then, with probability approaching 1 as n tends to ∞, G has the following properties:*

$$\begin{cases} \text{For every } X \subseteqq I \text{ of cardinality } |X| \leqq \alpha n,\ |N(X)| \geqq \beta|X|, \text{ and, similarly} \\ \text{for every } Y \subseteqq O \text{ of cardinality } |Y| \leqq \alpha n,\ |N(Y)| \geqq \beta|Y|. \end{cases} \tag{4.2}$$ $\blacksquare$

Note that in fact the result of [7] guarantees only the first half of (4.2) with probability $\to 1$ but by symmetry the second half can also be guaranted. Note also that we allow multiple edges here.

Suppose, now, that $\alpha=(1/2)+(\varepsilon/4)$ and $\beta=1+\varepsilon$, where $(1/2)+(\varepsilon/4)(1+\varepsilon)< <1$. It is easy to see that in this case (4.2) implies that G is a $(2n, d, \varepsilon)$-magnifier. Indeed, suppose $X \subseteqq I \cup O$, $|X| \leqq n$. Put $X_1 = X\cap I$, $X_2 = X \cap O$. If $X_1, X_2 \leqq \alpha n$ then $|N(X)-X| \geqq |\beta X_1|-|X_2|+|\beta X_2|-|X_1| = \varepsilon|X|$, as needed. Otherwise we can assume, without loss of generality, that $|X_1|>\alpha n=(1/2)+(\varepsilon/4)n$. But then $|X_2| \leqq \leqq (1/2)-(\varepsilon/4)n$ and

$$|N(X)-X| \geqq |N(X_1)-X_2| \geqq (1+\varepsilon)\left(\frac{1}{2}+\frac{\varepsilon}{4}\right)n - \left(\frac{1}{2}-\frac{\varepsilon}{4}\right)n > \varepsilon n \geqq \varepsilon|X|,$$

as needed. Thus G is a $(2n, d, \varepsilon)$-magnifier and by Lemma 2.4 $\lambda=\lambda(G) \geqq \varepsilon^2/(4+2\varepsilon^2)$. Therefore if d satisfies (4.1) with $\alpha\beta<1$, $\alpha=(1/2)+(\varepsilon/4)$, $\beta=1+\varepsilon$, and if n is large enough and G is chosen randomly as above, then almost certainly

$$\lambda(G) \geqq \varepsilon^2/(4+2\varepsilon^2). \tag{4.3}$$

This can be checked efficiently, by computing the eigenvalue λ. (In the unlikely case that (4.3) fails to occur, one can generate another G.) If (4.3) is satisfied, then, by Lemma 3.3, G is guaranteed to be an (n, d, c)-strong expander, where

$$c \geqq \frac{1}{d^2}(2d\lambda(G)-\lambda^2(G)) \geqq \frac{\varepsilon^2}{(4+2\varepsilon^2)d^2}\left(2d-\frac{\varepsilon^2}{4+2\varepsilon^2}\right). \tag{4.4}$$

Another possibility is to apply Corollary 2.3 to conclude that G is a $(2n, d, c')$ magnifier with $c'=2\lambda/(d+2\lambda)$ and then to apply Lemma 2.1 to construct from G its double cover that will be a (provably) $(2n, d+1, c')$ expander. If a higher amount of expansion is necessary one can use Proposition 2.6 to estimate the expanding properties of "iterates" of G.

We proceed to give one numerical example. Suppose $\varepsilon=1/8$, $\alpha=(1/2)+(\varepsilon/4)$, $\beta=1+\varepsilon$, $d=3$. One can easily check that these values satisfy (4.1). Hence, by the above discussion, almost every 3-regular bipartite graph G with n inputs and n outputs (chosen by 3 random permutations) is a $(2n, 3, 1/8)$-regular magnifier. If G is such a graph then, by (4.3), $\lambda(G)\geqq 1/258$, and thus, by (4.4), one can prove efficiently that G is an $(n, 3, 0.0025)$ strong expander. Similar examples for $d>3$ can also be given.

As a by-product of our methods we obtain some new results on the distribution of eigenvalues of the adjacency matrices of random d-regular graphs.

For fixed $0<p<1$, let G be a random graph on a set V of n vertices in which every edge is chosen, independently, with probability p. Recall that the adjacent matrix of G is the $n\times n$ binary matrix $A_G=(a_{uv})_{u,v\in V}$ in which $a_{uv}=1\Leftrightarrow uv$ is an edge. The eigenvalues of G are the eigenvalues of A_G. From the well known semicircle law of Wigner [33] it follows that with probability $1-o(1)$ (as $n\to\infty$) all but $o(n)$ of the eigenvalues of G have absolute value $<c\sqrt{n}$ for every $c>2\sqrt{p(1-p)}$. Füredi and Komlós [16] showed that in fact, with probability $\to 1$ as $n\to\infty$ the maximal eigenvalue is very close to $(n-2)p+1$ and all the others have absolute value $<2\sqrt{p(1-p)}\sqrt{n}+O(n^{1/3}\log n)$.

The situation becomes much more difficult for random d-regular graphs on n vertices, (fixed d, $n\to\infty$). A probabilistic model for these graphs was introduced in [10]. In [23] McKay determined the asymptotic behaviour of the eigenvalues thus providing an analogue of the semi-circle law for regular graphs. His results imply that with probability $1-o(1)$ all but $o(n)$ of the eigenvalues have absolute value $\leqq 2\sqrt{d-1}$. Note that the largest eigenvalue of a d-regular graph G is always d and the second largest is $d-\lambda(G)$, where $\lambda(G)$ is the second smallest eigenvalue of the Q-matrix of G. It seems unlikely that the methods of [16] can be extended to show that with high probability the second largest eigenvalue of a d-regular random graph G is well separated from the first. However, since this difference is $\lambda(G)$ and since one can easily check that with high probability G is a magnifier (provided $d\geqq 3$), Lemma 2.4 shows that the difference is almost always $\geqq\varepsilon(d)$, where $\varepsilon(d)$ is independent of the number of vertices. The following theorem can be proved using the ideas of the proof of Lemma 2.4.

Theorem 4.2. *Let G be a random d-regular graph on n vertices, (using the model of* [10]*), where $d \geqq 3$. Then, with probability $1-o(1)$ (as $n \to \infty$), $\lambda(G)$ = the difference between the largest and the second largest eigenvalue of G satisfies $\lambda(G) = \Omega(d)$.* ▮

We omit the detailed proof.

R. Boppana and the present author showed that for every d-regular graph G on n vertices $\lambda(G) \leqq d - 2\sqrt{d-1} + O(\log_d n)^{-1}$. (Note that by the results of [23] for a *random* d-regular graph on a large number n of vertices $\lambda(G) \leqq d - 2\sqrt{d-1}(1+o(1))$ with high probability.) It might be that for such graphs almost certainly $\lambda(G) \simeq \simeq d - 2\sqrt{d-1}$. If this is true, the random method for generating guaranteed expanders discussed in this section will supply much better expanders than the best known explicit construction described in [4, 5].

5. Concluding remarks

1. Further relations between the eigenvalue $\lambda(G)$ of a graph G and structural properties of G (diameter, bisection width, etc.) appear in [3].

2. The following conjecture was mentioned at the end of the previous section.

Conjecture 5.1. If G is a random d-regular graph on n vertices then with probability $\to 1$ as $n \to \infty$ $\lambda(G) \geqq d - 2\sqrt{d-1} + o(1)$.

Any lower bound of the form $d - O(\sqrt{d})$ would also be interesting.

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Added in proof: Recently, Lubotzky, Phillips, and Sarnak [34] constructed, for every fixed $d = p+1$, p prime, an infinite family of d-regular graphs G with $\lambda(G) \geqq d - 2\sqrt{d-1}$.

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